Tutorial 2

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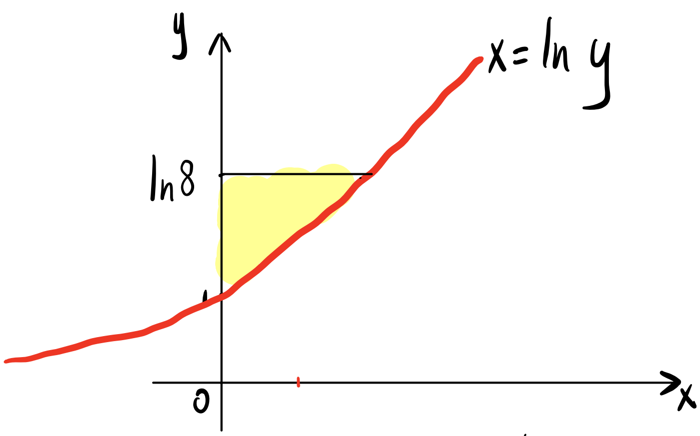
3rd week

1. Sketch the region of integration and evaluate the integral

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,\mathrm{d}x\mathrm{d}y$$



$$\text{Let } A(y) = \int_0^{\ln y} e^{x+y}\,dx$$

$$= e^y \int_0^{\ln y} e^x\,dx$$

$$= e^y \left[e^x\right]\Big|_0^{\ln y}$$

$$= e^y (y-1)$$

$$\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy = \int_1^{\ln 8} A(y)\,dy$$

$$= \int_1^{\ln 8} e^y (y-1)\,dy = \int_1^{\ln 8} (y-1)\,de^y$$

$$= e^y (y-1)\Big|_1^{\ln 8} - \int_1^{\ln 8} e^y\,d(y-1)$$

$$= 8(\ln 8 - 1) - 0 - e^y\Big|_1^{\ln 8}$$

$$= 8\ln 8 - 8 - (8-e)$$

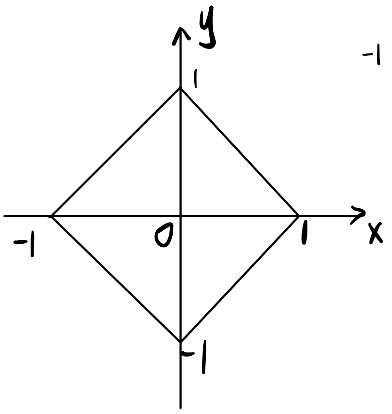
$$= 8\ln 8 + e - 16$$

2. Find the volume of the solid cut from the square column $|x| + |y| \leqslant 1$ by the plane $z = 0$ and $3x + z = 3$.

when $-1 \le x \le 0$ $\qquad$ $0 \le x \le 1$

$-1-x \le y \le 1+x$ $\qquad$ $-1+x \le y \le 1-x$

$z = 3-3x$ $\qquad$ $z = 3-3x$



$$V = \int_{-1}^{0}\int_{-1-x}^{1+x} 3-3x \, dy\, dx + \int_{0}^{1}\int_{-1+x}^{1-x} 3-3x \, dy\, dx$$

$$= \int_{-1}^{0} \left[(3-3x)y\right]\Big|_{y=-1-x}^{y=1+x} dx + \int_{0}^{1} \left[(3-3x)y\right]\Big|_{y=-1+x}^{y=1-x} dx$$

$$= \int_{-1}^{0} 2(3-3x)(1+x)\, dx + \int_{0}^{1} 2(3-3x)(1-x)\, dx$$

$$= \int_{-1}^{0} -6x^2+6\, dx + \int_{0}^{1} 6x^2-12x+6\, dx$$

$$= \left[-2x^3+6x\right]\Big|_{-1}^{0} + \left[2x^3-6x^2+6x\right]\Big|_{0}^{1}$$

$$= -(2-6) + [2-0].$$

$$= 6$$

3. Evaluate the improper integral

$$\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}} \mathrm{d}y\mathrm{d}x$$

Let $A(x) = \int_0^3 \frac{x^2}{(y-1)^{2/3}} dy$

$$A(x) = \int_0^1 \frac{x^2}{(y-1)^{2/3}} dy + \int_1^3 \frac{x^2}{(y-1)^{2/3}} dy$$

$$= \lim_{a \to 1^-} \int_0^a \frac{x^2}{(y-1)^{2/3}} dy + \lim_{a \to 1^+} \int_a^3 \frac{x^2}{(y-1)^{2/3}} dy$$

$$= \lim_{a \to 1^-} \left[3x^2 (y-1)^{1/3}\right]\Big|_0^a + \lim_{a \to 1^+} \left[3x^2 (y-1)^{1/3}\right]\Big|_a^3$$

$$= \lim_{a \to 1^-} \left[3x^2 (a-1)^{1/3} - 3x^2(-1)^{1/3}\right]$$

$$+ \lim_{a \to 1^+} \left[3x^2 (3-1)^{1/3} - 3x^2(a-1)^{1/3}\right]$$

$$= 3x^2 + 3x^2\sqrt[3]{2}$$

$$= (3 + 3\sqrt[3]{2})x^2$$

So $\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}} dy dx$

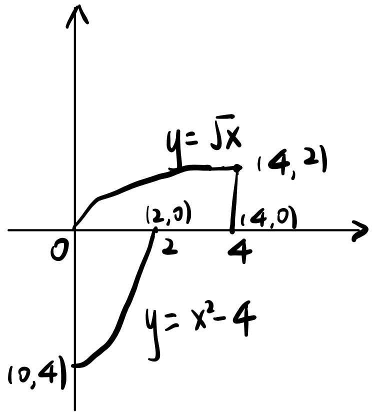
$$= \int_0^1 A(x) dx = \int_0^1 (3 + 3\sqrt[3]{2})x^2 dx$$

$$= (3 + 3\sqrt[3]{2}) \cdot \frac{1}{3} x^3 \Big|_0^1$$

$$= 1 + \sqrt[3]{2}$$

4. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

$$\int_0^2 \int_{x^2-4}^0 \mathrm{d}y\mathrm{d}x + \int_0^4 \int_0^{\sqrt{x}} \mathrm{d}y\mathrm{d}x$$



$$A_1 = \int_0^2 \int_{x^2-4}^0 dy\, dx$$

$$= \int_0^2 4 - x^2\, dx$$

$$= 4x - \frac{1}{3}x^3 \Big|_0^2$$

$$= \frac{16}{3}$$

$$A_2 = \int_0^4 \int_0^{\sqrt{x}} dy\, dx$$

$$= \int_0^4 \sqrt{x}\, dx$$

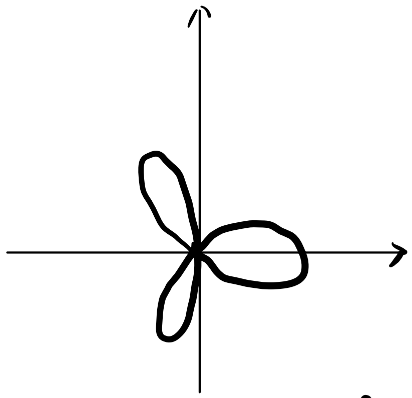
$$= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^4$$

$$= \frac{16}{3}$$

$$A = A_1 + A_2 = \frac{12}{3}$$

5. Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$

$$A = 6\int_0^{\frac{1}{6}\pi}\int_0^{12\cos 3\theta} r\,dr\,d\theta.$$



$$= 6\int_0^{\frac{1}{6}\pi} \frac{1}{2}r^2\Big|_{r=0}^{r=12\cos 3\theta}\,d\theta$$

$$= 6\int_0^{\frac{1}{6}\pi} 72\cos^2 3\theta\,d\theta$$

$$= 432\int_0^{\frac{1}{6}\pi}\left(\frac{1}{2}+\frac{1}{2}\cos 6\theta\right)d\theta$$

$$= 432\left(\frac{1}{2}\theta+\frac{1}{12}\sin 6\theta\right)\Big|_0^{\frac{1}{6}\pi}$$

$$= 432\left(\frac{1}{12}\pi + 0\right)$$

$$= 36\pi$$

$r = 12\cos 3\theta \geq 0$

$\theta \in [-\pi, \pi)$

$3\theta \in [-3\pi, 3\pi)$

So $-\frac{\pi}{2} \leq 3\theta \leq \frac{\pi}{2}$

or $-\frac{5\pi}{2} \leq 3\theta \leq -\frac{3\pi}{2}$

or $\frac{3\pi}{2} \leq 3\theta \leq \frac{5\pi}{2}$

So $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}] \cup [-\frac{5\pi}{6}, -\frac{\pi}{2}] \cup [\frac{\pi}{2}, \frac{5\pi}{6}]$